Quasi-invariant means and Zimmer amenability

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Abstract

Let Γ be a countable group acting on a countable set X by permutations. We give a necessary and sufficient condition for the action to have a quasi-invariant mean with a given cocycle. This can be viewed as a combinatorial analogue of the condition for the existence of a quasi-invariant measure in the Borel case given by Miller. Then we show a geometric condition that guarantees that the corresponding action on the Stone-Čech compactification is Zimmer amenable. The geometric condition (weighted hyperfiniteness) resembles Property A. We do not know the exact relation between the two notions, however, we can show that amenable groups and groups of finite asymptotic dimension are weighted hyperfinite.

1 Introduction

Quasi-invariant means. Let Γ be a countable group acting on a countable set X by permutations. Following John von Neumann we call a finitely additive probability measure on X an *invariant mean* μ if it is preserved by the action, that is,

$$\mu(gA) = \mu(A)$$

holds for any $g \in \Gamma$ and $A \subseteq X$. The existence of the invariant mean is equivalent to the existence of a Følner sequence $\{F_n\}_{n=1}^{\infty}$ having the following properties :

- $F_n \subset \Gamma, \quad |F_n|$ is finite for any $n \geq 1$.

- For any $\epsilon > 0$ and finite subset $L \subset \Gamma$ there exists a positive integer $n_{\epsilon,L}$ such that if $n \geq n_{\epsilon,L}$ then

$$\frac{|gF_n \cup F_n|}{|F_n|} < 1 + \epsilon$$

provided $g \in L$.

Recall that a group is *amenable* if the natural left action on itself possesses an invariant mean. Our first goal is to investigate quasi-invariant means, that is, finitely additive probability measures for which the zero measure class is preserved.

Before getting into details, let us consider the Borel analogue of the problem. Let Γ be a countable group acting by Borel automorphisms on a Polish space Y. A Borel probability measure ν on Y is an invariant measure if

$$\nu(gB) = \nu(B)$$

for any $g \in \Gamma$ and Borel set $B \subset Y$. Miller studied quasi-invariant measures given by a given cocycle ρ (see [8]) for definitions). Recall that a ρ-invariant mean ν satisfies

$$\nu(gB) = \int_B \rho(g, x) \, d\nu(x)$$

for any $g \in \Gamma$ and Borel set $B \subseteq Y$. Miller gave a necessary and sufficient condition for the existence of ρ-invariant means.

Now let us consider the discrete analogue of ρ-invariant means. Again, let Γ be a countable group acting on a countable set X by permutations and w be a positive real function on X. We further suppose that for any fixed $g \in \Gamma$ the function $x \to \frac{w(gx)}{w(x)}$ is bounded on X (later referred to as *boundedness condition*). First observe that $\rho(g, x) := \frac{w(gx)}{w(x)}$ is a Γ-cocycle on X. We say that a finitely additive probability measure μ on X is w-invariant if for any $g \in \Gamma$ and $A \subseteq X$

$$\mu(gA) = \int_A \frac{w(gx)}{w(x)} d\mu(x) \tag{1}$$

Recall [9] that any finitely additive probability measure μ extends to a bounded linear functional on $l^\infty(X)$, hence the integral notation is meaningful. Since the step functions are dense in $l^\infty(X)$, (1) can be reformulated the following way:

$$\int_X F(gx) \, d\mu(x) = \int_X \frac{w(g^{-1}x)}{w(x)} F(x) d\mu(x) \,, \tag{2}$$

where $F \in l^\infty(X), g \in \Gamma$. We will see that if Γ is any finitely generated group then there exists $w : \Gamma \to \mathbb{R}^+$ satisfying the boundedness condition such that w-invariant means exist on Γ for the natural left action.

Definition 1. *A family of finite subsets of X, $\{F_n\}_{n=1}^\infty$ forms a w-Følner sequence if for any $\epsilon > 0$ and finite set $L \subset \Gamma$ there exists $n_{\epsilon,L}$ such that if $n \geq n_{\epsilon,L}$ and $g \in L$ then*

$$\frac{\sum_{x \in gF_n \cup F_n} w(x)}{\sum_{x \in F_n} w(x)} < 1 + \epsilon \,.$$

Notice that if $w = 1$ then w-Følner sequences are exactly the usual Følner sequences. Our first theorem generalizes the classical result on the existence of invariant means.

Theorem 1. *Let Γ be a countable group acting on a countable set X by permutations. Let w be a positive function on X satisfying the boundedness condition. Then the following two conditions are equivalent.*

- *There exist w-invariant means.*

- *There exists a w-Følner sequence.*

Weighted hyperfinite graphs. Let G be a connected, infinite graph of bounded vertex degrees. We say that G is *weighted hyperfinite* if for any $\epsilon > 0$ there exists $K_\epsilon > 0$ with the following property. For any finite induced subgraph $L \subset G$ and nonnegative function $w : V(L) \to \mathbb{R}$ one can delete a subset M of vertices (together with all the incident edges) of L such that

- $\sum_{x \in M} w(x) \leq \epsilon \sum_{x \in V(L)} w(x)$

- All the remaining components have size at most K_ϵ.

In this case we say that M is a (w, ϵ)-*separating set* for L with component sizes at most K_ϵ. If H is a finite subset of the vertices we define $w(H) := \sum_{x \in H} w(x)$. We will call a positive function w *balanced* if there exists $C > 1$ such that

$$\frac{1}{C} < \frac{w(y)}{w(x)} < C$$

for any adjacent vertices x, y. Observe that the boundedness condition for w is equivalent to being balanced. We will show that being weighted hyperfinite is invariant under quasi-isometries. Hence, we can speak about weighted hyperfinite groups. We will prove the following theorem.

Theorem 2. *Let Γ be a finitely generated amenable group or a group of finite asymptotic dimension. Then Γ is weighted hyperfinite.*

Zimmer amenability. Let Γ be a finitely generated group with symmetric generating system S. Let Γ act on a compact Hausdorff space Y preserving the measure class of a Borel probability measure. That is, the action of Γ is quasi-invariant. The orbit equivalence relation E of the action is defined the following way: (xEy) is and only if x and y are in the same orbit, that is, $gx = y$ for some $g \in \Gamma$. The action is called *Zimmer amenable* [5],[11] if

- It is hyperfinite, that is, there exist finite Borel subrelations (all the equivalence classes are finite) $F_1 \subset F_2 \subset \ldots$ such that $\cup_{n=1}^{\infty} F_n = E$ modulo a zero class.

- Almost all the point stabilizers are amenable.

In this paper we will consider essentially free actions. In this case, hyperfiniteness and Zimmer amenability coincide.

Now let $\beta\Gamma$ be the Stone-Čech compactification of Γ. The elements of $\beta\Gamma$ are the ultrafilters on Γ. The principal ultrafilters are identified with the elements of Γ. A base of compact, Hausdorff topology on $\beta\Gamma$ is given by $\{A_*\}_{A \subset \Gamma}$, where A_* is the set of ultrafilters containing A. Then

- $A_* \cup B_* = (A \cup B)_*$

- $A_* \cap B_* = (A \cap B)_*$

- $\overline{A_*} = \overline{(A_*)}$

- $\emptyset_* = \emptyset$

- $\Gamma_* = \beta\Gamma$.

Note that the sets A_* are both closed and open. Since Γ acts on the ultrafilters of Γ, a continuous Γ-action is given on $\beta\Gamma$, where

$$g(A_*) = (gA)_*$$

holds for any $g \in \Gamma$ and $A \subset \Gamma$. Recall that the space of continuous functions on $\beta\Gamma$ can be identified with $l^\infty(\Gamma)$. If $F \in l^\infty(\Gamma)$, we denote by F_* the corresponding element in $C(\beta\Gamma)$. Then

- $(FG)_* = F_* G_*$

- $(F + G)_* = F_* + G_*$

- $F_* \circ g = (F \circ g)_*$

By the Riesz representation theorem, if μ is a finitely additive probability measure on Γ then

$$\phi(F_*) := \int_\Gamma F d\mu$$

defines a regular Borel probability measure $\widetilde{\mu}$ on $\beta\Gamma$ such that

$$\int_{\beta\Gamma} F_* d\widetilde{\mu} = \int_\Gamma F d\mu$$

for any bounded real function F on Γ.

Now let $w : \Gamma \to \mathbb{R}$ be a positive real function such that for any $g \in \Gamma$ the real function on Γ given by $x \to \frac{w(gx)}{w(x)}$ is bounded. Define the function $z \to \rho(g, z)$ in $C(\beta\Gamma)$ as $(x \to \frac{w(gx)}{w(x)})_*$. Then ρ is a Γ-cocycle on $\beta\Gamma$. That is,

$$\rho(gh, z) = \rho(g, hz)\rho(h, z).$$

Indeed, by the Γ-equivariance of the correspondance $F \to F_*$

- $z \to \rho(gh, z) = (x \to \frac{w(ghx)}{w(x)})_*$

- $z \to \rho(g, hz) = (x \to \frac{w(ghx)}{w(hx)})_*$

- $z \to \rho(h, x) = (x \to \frac{w(hx)}{w(x)})_*$

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Therefore

$$\int_{\beta\Gamma} F_*(gz)d\widetilde{\mu}(z) = \int_{\beta\Gamma} F_*(z)\rho(g^{-1}, z)d\widetilde{\mu}(z)\,.$$

That is, $\widetilde{\mu}$ is a ρ-invariant measure. In other words, the action of Γ is quasi-invariant on $\beta\Gamma$ with Radon-Nykodym cocyle ρ. The next result sheds some light on the relation between weighted hyperfiniteness and Zimmer amenability.

Theorem 3. *Let Γ be a finitely generated group, with a positive, balanced weight function w and a w-Følner sequence $\{F_n\}_{n=1}^{\infty}$. If Γ is weighted hyperfinite, then the corresponding Γ action on $\beta\Gamma$ with respect to the measure $\widetilde{\mu}_{F,\omega}$ is Zimmer amenable.*

Let us recall that a group Γ has Property A if and only if its canonical action on the Stone-Čech compactification is topologically amenable [6]. Also, by [2, Corollary 3.3.8] if ν is a quasi-invariant measure with respect to a free topologically amenable action then it is Zimmer amenable. Hence, if Γ is of Property A then the conclusion of Theorem 3 also holds. It is well-known that amenable groups as well as groups of finite asymptotic dimension have Property A (by Theorem 2 they are weighted hyperfinite as well). Finally, if Γ is finitely generated and contains an embedded expander sequence, then it cannot have Property A [10]. Clearly, w-hyperfinite groups cannot have imbedded expander sequences.

Question 1. *What is the relation between weighted hyperfiniteness and Property A? Does any of these properties imply the other?*

2 Quasi-invariant means

Let Γ be a finitely generated group acting on the countable set X with a symmetric generating set S. Let w be a positive, balanced real function on X.

Proposition 2.1. *If there exists a w-Følner sequence, then there exist w-invariant means, as well.*

Proof. Let $\{F_n\}_{n=1}^{\infty}$ be a w-Følner system and ω be a nonprincipal ultrafilter on $\mathbb{N}$. Let $\lim_{\omega} : l^{\infty}(X) \to \mathbb{R}$ be the corresponding ultralimit. Define μ by

$$\mu(A) := \lim_{\omega} \frac{\sum_{A \cap F_n} w(x)}{\sum_{x \in F_n} w(x)}\,.$$

Then μ is clearly a finitely additive measure extending to a mean on $l^{\infty}(X)$ by

$$\int_X F(x)d\mu(x) := \lim_{\omega} \frac{\sum_{x \in F_n} F(x)w(x)}{\sum_{x \in F_n} w(x)}\,.$$

Hence

$$\int_X F(gx)d\mu(x) = \lim_{\omega} \frac{\sum_{x \in F_n} F(gx)w(x)}{\sum_{x \in F_n} w(x)}\,.$$

By the w-Følner property

$$\lim_{n \to \infty} \frac{|\sum_{x \in F_n} F(gx)w(x) - \sum_{x \in F_n} F(x)w(g^{-1}x)|}{\sum_{x \in F_n} w(x)} = 0$$

holds for any $g \in \Gamma$.

Hence,

$$\int_X F(gx)d\mu(x) = \lim_\omega \frac{\sum_{x \in F_n} F(x)w(g^{-1}x)}{\sum_{x \in F_n} w(x)} = \int_X F(x)\frac{w(g^{-1}x)}{w(x)}d\mu(x).$$

$\square$

We will denote the invariant mean constructed above by $\mu_{F,\omega}$. Let $T \subset \Gamma$ be a finite set and $\{\Psi_g\}_{g \in T}$ be bounded positive functions on X. We say that $\{\Psi_g\}_{g \in T}$ is a w-compression system if for any $x \in X$

$$\sum_{g \in T} \Psi_g(x) = 1 \quad \text{and} \quad \sum_{g \in T} \Psi_g(g^{-1}x)\frac{w(g^{-1}x)}{w(x)} < \frac{1}{2}.$$

The notion of a w-compression system is motivated by Miller's idea of ρ-compressability [8].

Proposition 2.2. *If there exists a w-compression system then there exists no w-invariant mean.*

Proof. Suppose that μ is a w-invariant mean. Then

$$1 = \sum_{g \in T} \int_X \Psi_g(x)d\mu(x) = \sum_{g \in T} \int_X \Psi_g(g^{-1}x)\frac{w(g^{-1}x)}{w(x)}d\mu(x) < \frac{1}{2}$$

providing a contradiction. $\square$

Proposition 2.3. *If w-Følner systems do not exist, then we have a w-compression system.*

Proof. Let $a_g(x) := \Psi_g(x)w(x)$. Then for any $x \in X$

$$\sum_{g \in T} a_g(x) = w(x) \quad \text{and} \quad \sum_{g \in T} a_g(g^{-1}x) < \frac{1}{2}w(x). \tag{3}$$

Thus we need to prove that there exists a system $\{a_g\}_{g \in T}$ satisfying (3).

A classical application of the Max Cut-Min Flow Theorem is the Transportation Problem. Say, we have a finite bipartite graph $G = (V, E)$, where the vertex set V is the disjoint union of A and B. Let $p : V \to \mathbb{R}$ be a positive function. One can think about the elements x of set A as manufacturer producing a certain (divisible) product worth of $p(x)$. The elements y of B are buyers having $p(y)$ amount of money to spend. A manufacturer can sell goods only

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to the buyers he is connected to. The question (the Transportation Problem) is whether the manufacturer can sell all the goods to the buyers or not. In mathematical terms the problem is to associate nonnegative numbers $a(x, y)$ to the edges (x, y) such that

- For any $x \in A$
$$\sum_{(x,y)\in E} a(x, y) = p(x) \, .$$

-
$$\sum_{(x,y)\in E} a(x, y) \le p(y) \, .$$

According to the Max Cut Min Flow Theorem, the sufficient and necessary condition for the solvability of the transportation problem is that for any subset $L \subseteq A$
$$\sum_{x\in L} p(x) \le \sum_{y\in K} p(y) \, ,$$

where K is the set of vertices in Y adjacent to a vertex in X.
By compactness, the solvability of the Transportation Problem has the same necessary and sufficient condition even if G is an infinite bipartite graph with bounded vertex degrees. Let us see, how can we use the transportation problem. Suppose that there exists no w-Følner system. Then there exists a positive $\epsilon > 0$ such that for any finite subset $C \subset \Gamma$

$$(1 + \epsilon) \sum_{x\in C} w(x) < \sum_{y, \, y=gx, g\in S, x\in C} w(y) \, .$$

Hence there exists some $k > 0$ such that for any $x \in X$

$$2 \sum_{x\in C} w(x) < \sum_{y, \, y=gx, g\in S^k, x\in C} w(y) \tag{4}$$

doubling condition is satisfied. Let us remark that the idea of using doubling conditions is due to Deuber, Simonovits and Sós [4]. In their paper they used the Marriage Lemma, which is also a classical special case of the Max Flow Min Cut Theorem.
Now let us construct our bipartite graph G. Let both the left and the right vertex set of G be X. Draw an edge (x, y) if $y = g^{-1}x, g \in S^k := T$. For the vertices x on the left, define $p(x)$ to be $w(x)$. For the vertices y on the right, define $p(y)$ to be $\frac{1}{2}w(y)$. Then the equation (4) is just the necessary and sufficient condition of the corresponding Transportation Problem. Hence (3) can be satisfied. $\qquad\square$

By Propositions 2.1,2.2 and 2.3, Theorem 1 holds if Γ is finitely generated. Now let Γ be an arbitrary countable group acting on X and let w be a positive, balanced real function on X. Clearly, Γ has a w-Følner system if and only if all

of its finitely generated subgroups possesses a w-Følner system. Also, if there exists a w-invariant mean for each finitely generated subgroup then there exists a w-invariant mean for Γ as well.

Indeed, let $a_1, a_2, \ldots$ be an enumeration of the elements of Γ and $K_n \subset M_\Gamma$ be the set of invariant means with respect to the group generated by the set $\{a_1, a_2, \ldots, a_n\}$, where M_Γ is the compact Hausdorff space of all means on Γ [9].

Since $K_1 \supset K_2 \supset \ldots$ is a sequence of nonempty closed sets, there exists a w-invariant mean $\mu \in \cap_{n=1}^{\infty} K_n$. This finishes the proof of Theorem 1. $\qquad\square$

3 Weighted hyperfinite graphs

Proposition 3.1. *Weighted hyperfiniteness is invariant under quasi-isometries.*

Proof. Let G_1 and G_2 be quasi-isometric graphs with a uniform bound d on their vertex degrees, and suppose that G_2 is weighted hyperfinite. We need to show that G_1 is weighted hyperfinite as well. Let $\iota : G_1 \to G_2$ be a map that satisfies $c^{-1} d_{G_1}(x, y) - c \le d_{G_2}(\iota(x), \iota(y)) \le c d_{G_1}(x, y) + c$ with some $c > 0$ for any $x, y \in G_1$, and suppose that for every $z \in G_2$ there is some $x \in G_1$ such that $d_{G_2}(z, \iota(x)) \le c$ (note that slightly abusing notation we denote the graphs and their vertex sets by the same letters). In particular, for any $v \in V(G_2)$, $|\{x : \iota(x) = v\}| \le c^2$. Note that d^{2c+1} is an upper bound for the size of any ball of radius $2c$, and fix $C := \max\{d^{2c+1}, c^2\}$. Define a map $f : G_2 \to G_1$ as $f(z) := x$, where x is a point that minimizes $d_{G_2}(z, \iota(x)$ (fixed arbitrarily, in case of ambiguity). In particular, $\iota(f(z)) = z$ when $z \in \iota(G_1)$. We mention that every point of G_1 has at most C^2 preimages by f.

Let H_1 be an arbitrary finite induced subgraph of G_1; we want to show that for any $\epsilon > 0$ and any weight function w on H_1, there is a (w, ϵ)-separating set for H_1 with component sizes independent of the choice of H_1. Define $H_2 = \iota(H_1)$, and let H_2^+ be the $2c$-neighborhood of H_2. Define a weight function w' on H_2^+ to be $w'(z) := w(f(z))$. Then we have

$$w'(H_2^+) \le C^2 w(H_1) \tag{5}$$

by our observation on the number of preimages by f. Now define w'' on H_2^+ by letting $w''(z) := \sum_{y \in B(z)} w'(y)$, where $B(z)$ is the $2c$-neighborhood of z in H_2^+. We have noted that $|B(z)| \le C$, hence

$$w''(H_2^+) \le C w'(H_2^+) \le C^3 w(H_1), \tag{6}$$

using (5) for the second inequality.

Let S be an (w'', ϵ)-separating set for H_2^+ with components of sizes $K(\epsilon)$, and let S^+ be the $2c$-neighborhood of S in H_2^+. By definition of w'' we have

$$w'(S^+) \le w''(S). \tag{7}$$

We claim that $\iota^{-1}(S^+)$ is an $(w, C^3\epsilon)$-separating set for H_1 of component sizes $CK(\epsilon)$ (which would complete the proof, since ϵ was arbitrary, and C only depended on c and d). First, $w(\iota^{-1}(S^+)) \leq w'(S^+) \leq w''(S)$ using (7), and $w(H_1) \geq C^{-3}w''(H_2^+)$ by (6). Thus $w(\iota^{-1}(S^+))/w(H_1) \leq C^3 w''(S)/w''(H_2^+) \leq C^3\epsilon$. So it only remains to show that the components of $H_1 \setminus \iota^{-1}(S^+)$ have sizes at most $CK(\epsilon)$. This follows from the next claim.

Claim: Let $\iota(x), \iota(y) \in H_2$ be in different components of $H_2^+ \setminus S$. Then x and y are in different components of $H_1 \setminus \iota^{-1}(S^+)$.

Suppose not, and let P be a path between x and y in $H_1 \setminus \iota^{-1}(S^+)$. Consider $\iota(P)$. Since $P \cap \iota^{-1}(S^+) = \emptyset$, $\iota(P) \cap S^+ = \emptyset$. Hence $\iota(P)$ is at distance at least $2c$ from any element of S. On the other hand, two consecutive (adjacent) vertices u and v in P are mapped into points at distance at most $2c$ by the quasi-isometry ι. Therefore we can connect each such pair $\iota(u), \iota(v) \in H_2$ by a path of length at most $2c$ in H_2^+, which path is thus disjoint from S. The union of these paths between $\iota(u), \iota(v)$ in H_2^+ (over all such u, v) avoids S, hence $\iota(x)$ and $\iota(y)$ are in the same component of $H_2^+ \setminus S$. This contradicts the assumption on x and y, finishing the proof. $\qquad\square$

Proof of Theorem 2. Now let us prove that bounded degree graphs of finite asymptotic dimension are weighted hyperfinite. Recall that a graph G has asymptotic dimension d if for every $r > 0$ there exists an $R(r) = R$ and vertex-disjoint induced subgraphs $\mathcal{U}_1, \ldots, \mathcal{U}_d$ of G, such that every vertex of G is in some $\mathcal{U}_i$, for each $i \in \{1, \ldots, d\}$ every connected component of $\mathcal{U}_i$ has diameter at most R, and any two distinct components of $\mathcal{U}_i$ have distance at least r in G. If d is finite then we say that G has *finite asymptotic dimension*. The asymptotic dimension is a quasi-isometry invariant (hence it defines a group invariant). See e.g. [3] for a survey on the asymptotic dimension.

So, let d be the asymptotic dimension of the bounded degree graph G. Let d be the asymptotic dimension of G, H be an arbitrary induced subgraph of G, $w : V(H) \to \mathbb{R}$ be a weight function on the vertices, and $\epsilon > 0$. Define $r := 2[1 + 1/\epsilon]$, and let $\mathcal{U}_1, \ldots, \mathcal{U}_d$ be the families of sets corresponding to r in the definition of asymptotic dimension, and R be the corresponding $R(r)$. For $i \in \{1, \ldots, d\}$, $t \in \{1, 2, \ldots, 1 + [1/\epsilon]\}$, let $S_i(t) \subset V(G)$ be the set of points at distance t from $\mathcal{U}_i$. In particular, the sets $S_i(1), \ldots S_i(1 + [1/\epsilon])$ are pairwise disjoint. Hence there is a $t \in \{1, 2, \ldots, 1 + [1/\epsilon]\}$ such that $w(S_i(t)) \leq w(H)/(1 + [1/\epsilon]) \leq \epsilon w(H)$; let $t(i)$ be one such t. On the other hand, any two components of $\mathcal{U}_i$ are at distance at least $r = 2(1 + [1/\epsilon])$ from each other, thus any two such components are separated by $S_i(j)$ for any j. We obtain that $S := \cup_{i=1}^{d} S_i(t(i))$ is such a set that any component of $H \setminus S$ intersects at most one component of each U_i, hence its total diameter is at most $d(R+1)$. The uniform bound on the degrees of G then implies that every component of $H \setminus S$ has a uniformly bounded size. Finally, we have $w(S) = \sum_{i=1}^{d} w(S_i(t(i))) \leq d\epsilon w(H)$. Since ϵ was arbitrary, this shows that G is indeed hyperfinite.

Now we prove that the Cayley graph of a finitely generated amenable group

is weighted hyperfinite. Let Γ be a finitely generated amenable group with symmetric generating set S and G be its left-Cayley graph. That is the vertex set of G is Γ and the vertices x and y are connected if $x = sy$, for some $s \in S$. Let $\{F_n\}_{n=1}^{\infty}$ be a Følner sequence in Γ. For later convenience, suppose that $|\partial F_n|/|F_n| \leq n^{-2}$. Recall that ∂F_n is the set of elements in F_n that are connected to a vertex in the complement of F_n. Thinking about vertices of G as elements of the group, we will refer to products of vertices and vertex sets. Also, assume that the identity element is contained in each of the F_n. Note that for a subgraph H and $g \in \Gamma$, the map $x \mapsto xg$ from H preserves edges.

Let H be an arbitrary induced subgraph of G, and w a weight function on its vertices. Set $p_n \in [0,1]$ to be such that $(1 - p_n)^{|\partial F_n|} = 1 - n^{-1}$.

Define a random set R_n in G as follows: an $x \in V(G)$ will be in R_n with probability p_n and independently from the others. Define B_n to be $B_n = V(H) \setminus F_n R_n$. Finally, let $S_n \subset H$ be defined as $(\partial F_n) R_n \cap V(H)$. Any component of $H \setminus (B_n \cup S_n)$ has size at most $|F_n|$, because if $x \in V(H)$ is not in B_n, then there is some $v \in R_n$ with $v \in F_n^{-1} x$, and hence S_n separates x from any point in the complement of $F_n v$.

We claim that $B_n \cup S_n$ has relatively small weight for some choice of R_n. First, its expected weight is:

$$\mathbf{E}[w(S_n) + w(B_n)] = \mathbf{E}\big[\sum_{x \in H} w(x) \mathbf{1}_{x \in \partial F_n R_n}\big] + \mathbf{E}\big[\sum_{x \in H} w(x) \mathbf{1}_{x \notin F_n R_n}\big] =$$

$$= \sum_{x \in H} w(x)(\mathbf{P}[x \in \partial F_n R_n] + \mathbf{P}[x \notin F_n R_n]) =$$

$$= \sum_{x \in H} w(x)(\mathbf{P}[\partial F_n^{-1} x \cap R_n \neq \emptyset] + \mathbf{P}[F_n^{-1} x \cap R_n = \emptyset]) =$$

$$= w(H)(1 - (1-p)^{|\partial F_n|} + (1-p)^{|F_n|}) \leq w(H)(n^{-1} + e^{-n}),$$

where the last inequality follows from the assumption on F_n and the choice of p. Hence, there is some R_n where the corresponding S_n, B_n satisfies $w(S_n \cup B_n) \leq 2n^{-1} w(H)$. We have also observed that $S_n \cup B_n$ splits H into pieces of sizes at most $|F_n|$. Since H was arbitrary, we have proved that G is hyperfinite. $\square$

4   The Proof of Theorem 3

In this section we use some ideas from [1]. Let Γ be a finitely generated group with symmetric generating system S and let G be the associated (left) Cayley graph. The graphing $\mathcal{G}$ (see [7]) of the associated Γ-action on $\beta\Gamma$ is defined as follows: $x, y \in \beta\Gamma$ are connected if there exists $s \in S$ such that $sx = y$.

Lemma 4.1. *The action of Γ on $\beta\Gamma$ is free. Hence the components of $\mathcal{G}$ are isomorphic to G.*

Proof. Let $g \in \Gamma$ and $\omega \in \beta\Gamma$ be an ultrafilter. Then $g\omega$ is the ultrafilter containing the sets gA, where $A \in \omega$. Let $\cup_{i=1}^n A^i = \Gamma$ be a finite partition such that $gA^i \cap A^i = \emptyset$ for a fixed $g \in \Gamma$ and $1 \leq i \leq n$. Then $g(A^i_*) \cap A^i_* = \emptyset$. Since $\beta\Gamma = \cup_{i=1}^n A^i_*$ g cannot fix any element of $\beta\Gamma$. $\square$

Let $T \subseteq G$ be a subgraph such that the vertex set of T is the whole Γ. We can associate a Borel subgraphing $\mathcal{G}(T) \subset \mathcal{G}$ to T the following way. For $s \in S$, let $A_s \subset \Gamma$ be the set of vertices x such that x and sx are adjacent in T. Now connect $y \in (A_s)_*$ to sy. Hence the subgraphing $\mathcal{G}(T)$ is the union of the graphs of the Borel automorphisms $(A_s)_* \to s(A_s)_*$, where $s \in S$. Note that by a Borel subgraphing (as in [7]) we always mean a Borel subgraph of the graphing, such that the vertex set is the whole space Y. If $T \subset G$ let ∂T be the set of vertices x such that x is adjacent to some y, that is, not in the same T-component as x. Similarly, we can define $\partial(\mathcal{G}(T))$.

Proposition 4.1. *Let Γ, w and $\{F_n\}_{n=1}^\infty$ be as in Theorem 3. Then for any $\epsilon > 0$ there exists a subgraph $T_\epsilon \subset G$ with components of bounded size such that*

$$\mu_{F,\omega}(\partial T_\epsilon) < \epsilon. \tag{8}$$

Proof. Since $\{F_n\}_{n=1}^\infty$ is w-Følner, for any $\delta > 0$ we have a subgraph F_n^δ, $\frac{w(F_n^\delta)}{w(F_n)} < \delta$, such that if we delete F_n^δ from F_n the components of the resulting graph are bounded by K_δ. Let T_δ be the union of these components plus all the points outside the union of the w-Følner sets as singletons. That is the vertex set of T_δ is Γ. Clearly, if $x \in \partial T_\delta \cap F_n$ then either $x \in \partial F_n$ of x is in the 1-neighborhood of F_n^δ, $B_1(F_n^\delta)$. By the Følner property,

$$\lim_{n \to \infty} \frac{w(\partial F_n)}{w(F_n)} = 0.$$

Also,

$$w(B_1(F_n^\delta)) \leq C|S|w(\partial F_n),$$

where C is the constant in the balancedness condition for w. Therefore, $\mu_{F,\omega}(\partial T_\epsilon) \leq C|S|\delta$. Hence if $\delta < \epsilon/C|S|$ the equation (8) is satisfied. $\square$

Lemma 4.2. *Let $T \subset G$ be a subgraph such that all the components of T have size at most k. Then all the components of $\mathcal{G}(T)$ have size at most k as well. Moreover, $\partial(\mathcal{G}(T)) = (\partial T)_*$.*

Proof. Let $A \subseteq \Gamma$ be a set containing exactly one element from each component. For each subgraph of G we have a set $A_F \subset A$ of vertices x such that the component of x is isomorpic to Fa (even as a graph with edge labels from S). Then

$$\mathcal{G}(T) = \cup_F F(A_F)_*, \tag{9}$$

hence all the components of $\mathcal{G}(T)$ have size at most k. For the second statement,

$$\partial(\mathcal{G}(T)) = \partial(\cup_F F(A_F)_*) = \cup_F \partial(F(A_F)_*) = \cup_F (\partial F)(A_F)_* = (\partial T)_*$$

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□

By [7, Proposition 10.3], $\mathcal{G}$ is hyperfinite if and only if for any $\epsilon > 0$ there exists a Borel subgraphing $\mathcal{S}_\epsilon \subset \mathcal{G}$ such that

$$\widetilde{\mu}_{F,\omega}(\partial \mathcal{S}_\epsilon) \leq \epsilon \tag{10}$$

Let $\mathcal{S}_e := (T_\epsilon)_*$, then (10) follows. This ends the proof of Theorem 3.   □

Finally, let us show that for any finitely generated group Γ, there exists a positive, balanced function $w : \Gamma \to \mathbb{R}$ such that:

- there exist w-Følner systems,

- the resulting measure $\widetilde{\mu}_{F,\omega}$ is atomless.

First of all, we can suppose that Γ is nonamenable, since for amenable groups $w := 1$ clearly satisfies the two conditions. Let $\{B_r(x_r)\}_{r=1}^{\infty}$ be vertex disjoint balls in a Cayley-graph G of Γ. Note that the distance d in G is the shortest path metric and $B_r(x_r)$ is the r-ball around x_r. Let w_r be defined on $B_r(x_r)$ the following way. For any $0 \leq i \leq r$, $w_r(S_i(x_r)) = \frac{1}{r+1}$, and $w_r(x) = w_r(y)$ if $x, y \in S_i(x_r)$, where

$$S_i(x_r) = \{y \mid d(x_r, y) = i\}.$$

Then, for $x \in B_r(x_r)$ let $w(x) = \frac{w_r(x)}{w_r(z)}$, where $z \in S_r(x_r)$. If $x \notin \cup_{r=1}^{\infty} B_r(x_r)$, let $w(x) = 1$. Then w is balanced, since by nonamenability, and by the fact that Γ is finitely generated, there exists $D > 1$ such that for all $r \geq 1$ and $0 \leq i \leq r$,

$$\frac{1}{D} < \frac{|S_{i+1}(x_r)|}{|S_i(x_r)|} < D$$

Clearly, $\{B_r(x_r)\}_{r=1}^{\infty}$ forms a w-Følner system. Observe that for any $k \geq 1$ there exists $r_k \geq 1$ such that if $r \geq r_k$ then we can partition $B_r(x_r)$ into k parts such that the weight of each part is less than $\frac{2}{k} w(B_r(x_r))$. This observation easily follows from the fact that

$$\lim_{r \to \infty} \frac{\max_{y \in B_r(x_r)} w(y)}{w(B_r(x_r))} = 0.$$

Therefore, for any $\delta > 0$ one can partition $\beta\Gamma$ into finitely many Borel parts such that the $\widetilde{\mu}_{F,\omega}$-measure of each part is less than δ. Hence $\widetilde{\mu}_{F,\omega}$ is atomless.

References

[1] M. ABÉRT AND G. ELEK, Hyperfinite actions on countable sets and probability measure spaces. *to appear* http://arxiv.org/pdf/1105.3200v3

[2] C. ANANTHARAMAN-DELAROCHE AND J. RENAULT, Amenable groupoids. *Monographies de L'Enseignement Mathématique* **36** (2000)

[3] D. BELL AND A. DRANISHNIKOV , Asymptotic dimension. *Topology Appl.* **155** (2008) no. 12, 1265–1296.

[4] W. A. DEUBER, M. SIMONOVITS AND V. T. SÓS, A note on paradoxical metric spaces. *Studia Sci. Math. Hung.* **30** (1995), no. 1-2, 17-23.

[5] G. A. ELLIOTT AND T. GIORDANO, Amenable actions of discrete groups. *Ergodic Theory Dynam. Syst.* **13** (1993) no.2, 289-318.

[6] N. HIGSON AND J. ROE, Amenable group actions and the Novikov conjecture. *J. Reine. Angew. Math* **519** (2000) 143-153.

[7] A. KECHRIS AND B. D. MILLER, Topics in orbit equivalence theory. *Lecture Notes in Mathematics* **1852** Springer Verlag, 2004.

[8] B. D. MILLER, The existence of measures of a given cocycle. II. Probability measures. *Lecture Notes in Mathematics* **28** (2008) no.5, 1613-1633.

[9] A. L. T. PATERSON, Amenability. *Mathematical Surveys and Monographs* **29** (1988) American Mathematical Society.

[10] J. ROE, Lectures on coarse geometry. *University Lecture Series*, **31** American Mathematical Society, Providence, RI, 2003.

[11] R. ZIMMER, Amenable ergodic group actions and an application to Poisson boundaries of random walks. *Journal of Functional Analysis* **27** (1978), no. 3, 350-372.

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